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# *INTEGRATION BY AUXILIARY INTEGRALS.*

BY WERNER A. STILLE, PH. D., HIGHLAND, ILL.

WHEN the integration of a function of the form $dy = f(x)\, dx$ is required in finite terms, we must reduce the function to some one or more of the well known fundamental forms. By far the most fruitful method of so doing is the substitution of a new variable; and the real difficulty to be overcome consists in the judicious choice of the form of the new variable in terms of $x$. Very often the reduction to any of the fundamental forms is found impossible, or at least all our attempts to the purpose prove futile and we have no resource left but the expansion of the function into a converging series of which the single terms are then to be integrated.

It would seem at first sight that, although the integration in finite terms may be found impracticable, yet the opposite mode of procedure must yield an abundance of new results and must lead to new fundamental forms, viz., the assumption of some new form of the variable instead of the variable of the fundamental integrals, which subst. would immediately yield new and useful primitive integrals which might in their turn be used as we now use the primitive forms of the text-books. The great obstacle, however, to the invention of new primitive integrals in this manner lies in the fact that upon differentiating the new variable, the differential is commonly found to consist of more than one term, so that, although the aggregate value of the integrals so arising is known, yet the values of the integrals each singly, are not known.

The result of this state of affairs has been that to the present day the number of differentials integrable in finite terms is quite limited. This is especially true of the so-called irrational functions; and an examination of any collection of tables of integrals reveals the fact that of irrational functions scarcely any are integrable in finite terms save such where the variable occurs at no higher power than the second. Indeed besides these radicals of the second degree scarcely any but the "binomial" differentials seem to have been studied. I purposely leave out of consideration the elliptic integrals and other transcendental functions, as not falling under the forms proposed, namely those integrable in finite form; my intention being, in the present paper, to treat of an extension of the well known methods of reduction of differentials to the primitive forms by means of other differentials of which the integrals are already known.

In order to give, at the outset, some idea of the method here chiefly to be followed, I will illustrate the same by the following simple example. Let it be required to integrate

$$dy = \frac{dx}{(1+x^2)\sqrt{(1-x^2)}}.$$

This form readily suggests two of the primitive forms, namely:

$$\frac{dx}{1+x^2} \text{ and } \frac{dx}{\sqrt{(1-x^2)}}.$$

We will put

$$dz = \frac{dx}{1+x^2}, \text{ so that } z = \operatorname{arc}\operatorname{tg} x + C.$$

Since we are at liberty to assign to $z$ any value compatible with the last equation, we can put $C = 0$, so that

$$z = \operatorname{arc}\operatorname{tg} x, \quad \therefore x = \operatorname{tg} z.$$

We will now compare $dy$ with $dz$, thus

$$\frac{dy}{dz} = \frac{1}{\sqrt{(1-x^2)}} = \frac{1}{\sqrt{(1-\operatorname{tg}^2 z)}};$$

$$\therefore dy = \frac{dz}{\sqrt{(1-\operatorname{tg}^2 z)}} = \frac{\cos z . dz}{\sqrt{(1-2.\sin^2 z)}}, \text{ hence}$$

$$y = \tfrac{1}{2}\sqrt{2}.\operatorname{arc}\sin(\sqrt{2}.\sin z) + C.$$

On the other hand, we might also have integrated in this way: Putting $dt = dx \div \sqrt{(1-x^2)}$, $\therefore t = \operatorname{arc}\sin x$, $\therefore x = \sin t$.

$$\frac{dy}{dt} = \frac{1}{1+x^2} = \frac{1}{1+\sin^2 t}, \; dy = \frac{dt}{1+\sin^2 t};$$

$$\therefore y = \tfrac{1}{2}\sqrt{2}.\operatorname{arc}\operatorname{tg}(\sqrt{2}.\operatorname{tg} t) + C.$$

Now, in order to avail ourselves of the method just illustrated we will first extend somewhat the range of the primitive integrals of the books and then make use of the functions so found for the purpose just indicated. It will be seen in the sequel that we can easily by our method find a great number of integrals which to find in finite terms by the methods of the books would be impracticable.

For the sake of convenience I will arrange the new primitive functions into classes, or types, each corresponding to one of the fundamental forms of the text-books. I would beg leave, for this purpose, to retain the continental notation, namely arc sin $x$ for $\sin^{-1}x$ and arc tg $x$ for $\operatorname{tang}^{-1}x$ &c., which notation, it will be seen, is more suitable for our purpose.

## A. THE TYPE ARC SIN X.

The following integrals are easily verified by differentiation. The constant of integration is omitted. The inversion of the integral function is subjoined in each case, since with it we shall chiefly have to deal.

$$\int\frac{dx}{\sqrt{x}\sqrt{(1-x)}} = \frac{2}{1}\text{arc}\sin x^{\frac{1}{2}} = y;\ \therefore x = \left(\sin\frac{y}{2}\right)^{2}.$$

$$\int\frac{dx}{\sqrt{(1-x^{2})}} = \frac{2}{2}\text{arc}\sin x = y;\ \therefore x = \left(\sin\frac{2}{2}y\right)^{\frac{2}{2}} = \sin y.$$

$$\int\frac{\sqrt{x}.dx}{\sqrt{(1-x^{3})}} = \frac{2}{3}\text{arc}\sin x^{\frac{3}{2}} = y;\ \therefore x = \left(\sin\frac{3}{2}y\right)^{\frac{2}{3}}.$$

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$$\int\frac{x^{\frac{1}{2}n-1}dx}{\sqrt{(1-x^{n})}} = \frac{2}{n}\text{arc}\sin x^{\frac{1}{2}n} = y;\ \therefore x = \left(\sin\frac{n}{2}y\right)^{\frac{2}{n}};$$

where $n$ is any positive or negative integer or fraction.

Similarly:

$$\int\frac{dx}{x\sqrt{(x-1)}} = -2\,\text{arc}\sin x^{-\frac{1}{2}} = y;\ \therefore x = \frac{1}{(\sin\frac{1}{2}y)^{2}}.$$

$$\int\frac{dx}{x\sqrt{(x^{2}-1)}} = -\frac{2}{2}\text{arc}\sin x^{-1} = y;\ \therefore x = \frac{1}{(\sin y)}.$$

$$\int\frac{dx}{x\sqrt{(x^{3}-1)}} = -\frac{2}{3}\text{arc}\sin x^{-\frac{3}{2}} = y;\ \therefore x = \frac{1}{(\sin\frac{3}{2}y)^{\frac{2}{3}}}.$$

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$$\int\frac{dx}{x\sqrt{(x^{n}-1)}} = -\frac{2}{n}\text{arc}\sin x^{-\frac{1}{2}n} = y;\ \therefore x = \frac{1}{(\sin\frac{n}{2}y)^{\frac{2}{n}}};$$

where $n$ may be any positive or negative integer or fraction.

Similarly, when putting (as is customary) $i = \sqrt{-1}$,

$$\int\frac{dx}{\sqrt{x}\sqrt{(1+x)}} = -2i.\text{arc}\sin(i.x^{\frac{1}{2}}) = y;\ \therefore x^{\frac{1}{2}} = -i\sin\tfrac{1}{2}iy.$$

$$\int\frac{dx}{\sqrt{(1+x^{2})}} = -\frac{2}{2}i.\text{arc}\sin(i.x) = y;\ \therefore x = -i\sin iy.$$

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$$\int\frac{x^{\frac{1}{2}n-1}dx}{\sqrt{(1+x^{n})}} = -\frac{2}{n}i.\text{arc}\sin(i.x^{\frac{1}{2}n}) = y;\ \therefore x^{\frac{1}{2}n} = -i\sin\tfrac{n}{2}iy.$$

And so also:

$$\int\frac{dx}{x\sqrt{(x+1)}} = 2i.\text{arc}\sin(i.x^{-\frac{1}{2}}) = y;\ \therefore x^{-\frac{1}{2}} = i.\sin(i.\tfrac{1}{2}y).$$

$$\int\frac{dx}{x\sqrt{(x^{2}+1)}} = \frac{2}{2}i.\text{arc}\sin(i.x^{-1}) = y;\ \therefore x^{-1} = i.\sin(i.\ y).$$

$$\int\frac{dx}{x\sqrt{(x^{3}+1)}} = \frac{2}{3}i.\text{arc}\sin(i.x^{-\frac{3}{2}}) = y;\ \therefore x^{-\frac{2}{3}} = i.\sin(i.\tfrac{2}{3}y).$$

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$$\int\frac{dx}{x\sqrt{(x^{n}+1)}} = \frac{2}{n}i.\text{arc}\sin(i.x^{-\frac{1}{2}n}) = y;\ \therefore x^{-\frac{2}{n}} = i.\sin(i.\tfrac{n}{2}y).$$

The following forms belonging to the same type are also easily verified by differentiation:

$$\int\frac{dx}{(1+x)\sqrt{x}\sqrt{(1-x)}}=-\sqrt{2}.\text{arc sin}\sqrt{\frac{1-x}{1+x}}=y;\ \therefore\ x=\frac{1-\sin^2(\frac{1}{2}y\sqrt{2})}{1+\sin^2(\frac{1}{2}y\sqrt{2})}$$

$$\int\frac{dx}{(1+x^2)\sqrt{(1-x^2)}}=-\tfrac{1}{2}\sqrt{2}.\text{arc sin}\sqrt{\frac{1-x^2}{1+x^2}}=y;\ \therefore\ x^2=\frac{1-\sin^2(\ y\sqrt{2})}{1+\sin^2(\ y\sqrt{2})}$$

$$\int\frac{\sqrt{x}.dx}{(1+x^3)\sqrt{(1-x^3)}}=-\tfrac{1}{3}\sqrt{2}.\text{arc sin}\sqrt{\frac{1-x^3}{1+x_3}}=y;\ \therefore\ x^3=\frac{1-\sin^2(\frac{3}{2}y\sqrt{2})}{1+\sin^2(\frac{3}{2}y\sqrt{2})}$$

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$$\int\frac{x^{\frac{1}{2}n-1}dx}{(1+x^n)\sqrt{(1-x^n)}}=-\tfrac{1}{n}\sqrt{2}.\text{arc sin}\sqrt{\frac{1-x^n}{1+x^n}}=y;\ \therefore\ x^n=\frac{1-\sin^2(\frac{n}{2}y\sqrt{2})}{1+\sin^2(\frac{n}{2}y\sqrt{2})}$$

Again:

$$\int\frac{dx}{(1-x)\sqrt{x}\sqrt{(1+x)}}=-\sqrt{-2}.\text{arc sin}\sqrt{\frac{1+x}{1-x}}=y;\ \therefore\ x=\frac{\sin^2(\frac{1}{2}y\sqrt{-2})-1}{\sin^2(\frac{1}{2}y\sqrt{-2})+1}.$$

$$\int\frac{dx}{(1-x^2)\sqrt{(1+x^2)}}=-\tfrac{1}{2}\sqrt{-2}.\text{arc sin}\sqrt{\frac{1+x^2}{1-x^2}}=y;\ \therefore\ x^2=\frac{\sin^2(\ y\sqrt{-2})-1}{\sin^2(\ y\sqrt{-2})+1}$$

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$$\int\frac{x^{\frac{1}{2}n-1}dx}{(1-x^n)\sqrt{(1+x^n)}}=-\tfrac{1}{n}\sqrt{-2}.\text{arc sin}\sqrt{\frac{1+x^n}{1-x^n}}=y;\ \therefore\ x^n=\frac{\sin^2(\frac{1}{n}y\sqrt{-2})-1}{\sin^2(\frac{1}{n}y\sqrt{-2})+1}.$$

Again:

$$\int\frac{dx}{\sqrt{(2ax-x^2)}}=\text{arc sin}\frac{x-a}{a}=y;\ \therefore\ x=a(1+\sin\ y).$$

$$\int\frac{dx}{\sqrt{(2a-x^2)}}=\tfrac{1}{2}\text{arc sin}\frac{x^2-a}{a}=y;\ \therefore\ x^2=a(1+\sin 2y).$$

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$$\int\frac{x^{\frac{1}{2}n-1}dx}{\sqrt{(2a-x^n)}}=\tfrac{1}{n}.\text{arc sin}\frac{x^n-a}{a}=y;\ \therefore\ x^n=a(1+\sin ny).$$

And again:

$$\int\frac{dx}{x\sqrt{(2x-1)}}=\text{arc sin}\left(1-\frac{1}{x}\right)=y;\ \therefore\ x=\frac{1}{1-\sin\ y}.$$

$$\int\frac{dx}{x\sqrt{(2x^2-1)}}=\tfrac{1}{2}\text{arc sin}\left(1-\frac{1}{x^2}\right)=y;\ \therefore\ x^2=\frac{1}{1-\sin 2y}.$$

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$$\int\frac{dx}{x\sqrt{(2x^n-1)}}=\tfrac{1}{n}\text{arc sin}\left(1-\frac{1}{x^n}\right)=y;\ \therefore\ x^n=\frac{1}{1-\sin ny}.$$

## B. THE TYPE ARC TG X.

$$\int\frac{dx}{(2+x)\sqrt{(1+x)}} = 2\,\mathrm{arc\,tg}\sqrt{(1+x)} = y;\ \therefore x = \mathrm{tg}^2(\tfrac{1}{2}y)-1.$$

$$\int\frac{x^{n-1}dx}{(2+x^n)\sqrt{(1+x^n)}} = \frac{2}{n}\,\mathrm{arc\,tg}\sqrt{(1+x^n)} = y;\ \therefore x^n = \mathrm{tg}^2(\tfrac{1}{n}y)-1.$$

$$\int\frac{dx}{(2x+1)\sqrt{x}\sqrt{(x+1)}} = -\ \mathrm{arc\,tg}\left(1+\frac{1}{x}\right)^{\frac{1}{2}} = y;\ \therefore x = \frac{1}{\mathrm{tg}^2\,y-1}.$$

$$\int\frac{dx}{(2x^2+1)\sqrt{(x^2+1)}} = -\tfrac{1}{2}\mathrm{arc\,tg}\left(1+\frac{1}{x^2}\right)^{\frac{1}{2}} = y;\ \therefore x = \frac{1}{\mathrm{tg}^2 2y-1}.$$

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$$\int\frac{x^{\frac{1}{2}n-1}\,dx}{(2x^n+1)\sqrt{(x^n+1)}} = -\tfrac{1}{n}\mathrm{arc\,tg}\left(1+\frac{1}{x^n}\right)^{\frac{1}{2}} = y;\ \therefore x^n = \frac{1}{\mathrm{tg}^2 ny-1}.$$

## C. THE TYPE LOG X.

The following primitive integrals are also easily verified by differentiation.

$$\int\frac{dx}{x(x-1)} = \mathrm{lg}\left(1-\frac{1}{x}\right) = y;\ \therefore x = \frac{1}{1-e^y}.$$

$$\int\frac{dx}{x(x^2-1)} = \tfrac{1}{2}\mathrm{lg}\left(1-\frac{1}{x^2}\right) = y;\ \therefore x = \frac{1}{1-e^{2y}}.$$

$$\int\frac{dx}{x(x^3-1)} = \tfrac{1}{3}\mathrm{lg}\left(1-\frac{1}{x^3}\right) = y;\ \therefore x = \frac{1}{1-e^{3y}}.$$

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$$\int\frac{dx}{x(x^n-2)} = \tfrac{1}{n}\mathrm{lg}\left(1-\frac{1}{x^n}\right) = y;\ \therefore x = \frac{1}{1-e^{ny}}.$$

$$\int\frac{dx}{x(x+1)} = -\ \mathrm{lg}\left(1+\frac{1}{x}\right) = y;\ \therefore x = \frac{e^y}{1-e^y}.$$

$$\int\frac{dx}{x(x^2+1)} = -\tfrac{1}{2}\mathrm{lg}\left(1+\frac{1}{x^2}\right) = y;\ \therefore x = \frac{e^{2y}}{1-e^{2y}}.$$

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$$\int\frac{dx}{x(x^n+1)} = -\tfrac{1}{n}\mathrm{lg}\left(1+\frac{1}{x^n}\right) = y;\ \therefore x = \frac{e^{ny}}{1-e^{ny}}.$$

$$\int\frac{dx}{x\sqrt{(x^n-1)}\,[x^{\frac{1}{2}n}+\sqrt{(x^n-1)}]} = \tfrac{2}{n}\mathrm{lg}\left\{1+\left(1-\frac{1}{x^n}\right)^{\frac{1}{2}}\right\} = y;\ \therefore$$

$$x^n = \frac{1}{e^{\frac{1}{2}ny}(2-e^{\frac{1}{2}ny})}.$$

$$\int\frac{dx}{x\sqrt{(x^n+1)}\,[x^{\frac{1}{2}n}+\sqrt{(x^n-1)}]} = -\tfrac{2}{n}\mathrm{lg}\left\{1+\left(1+\frac{1}{x^n}\right)^{\frac{1}{2}}\right\} = y;\ \therefore$$

$$x^n = \frac{e^{ny}}{1-2e^{ny}}.$$

The application of the foregoing formulæ I begin with two functions commonly treated in the text-books. This will lead to a comparison of methods. Next I shall proceed to the integration of functions not found in the books.

The function

$$dy = \frac{x^m dx}{\sqrt{(1-x^2)}}$$

is usually treated in the books by means of a formula of reduction and the two cases where $m$ is odd or even are separately considered. This function at once calls to mind the elementary form

$$dz = \frac{dx}{\sqrt{(1-x^2)}},$$

from which $z = \text{arc} \sin x$; $x = \sin z$. Hence

$$\frac{dy}{dz} = x^m = \sin^m z;\ \therefore\ dy = \sin^m z \, dz.$$

First let $m$ be an odd number, for example $m = 5$, then

$$dy = \sin z\,(1-\cos^2 z)^2 dz = (1-2\cos^2 z+\cos^4 z) \sin z \, dz;$$

$$y = -\cos z + \tfrac{2}{3}\cos^3 z + \tfrac{1}{5}\cos^5 z.$$

This mode of treatment of $\int \sin^m z \, dz$ evidently applies to all cases where $m$ is odd. Now let $m$ be an even number, for example $m = 4$, then

$$dy = \sin^4 z \, dz = \tfrac{1}{4}(1-\cos 2z)^2 dz = \tfrac{1}{4}(1-2\cos 2z+\cos^2 2z) dz,$$

and expanding $\cos^2 2z$ as before,

$$dy = \tfrac{1}{4}(\tfrac{3}{4}-2\cos 2z+\tfrac{1}{2}\cos 4z) \, dz,$$

the integration of which is obvious. The method just employed for $m$=an even number is applicable to all cases of that kind. And let it be noticed by the way that all integrals of the forms

$$\int \cos^n x \, dx, \quad \int \sin^n x \, dx; \quad \int \sin^m x \cos^n x \, dx$$

may be found with equal facility as the above and without recourse to formulæ of reduction when $m$ and $n$ are positive. And I would mention also, in passing, that the same simple methods are still applicable in $\int \sin^m x \, dx$ and $\int \cos^m x \, dx$ when $m$ is negative. For when, in this case, $m$ is odd, the multiplication of numerator and denominator by $\cos x$ and by $\sin x$ respectively, readily suggests the further proceeding; and when $m$ is an even number the passage to the cognate imaginary angles by the substitution $\sin x = -i \operatorname{tg} i\varphi$, $\therefore \cos x = 1 \div \cos i\varphi$, $\therefore dx = d\varphi \div \cos i\varphi$ reduces this case to that where $m$ is odd, so that the formula of reduction given in the books becomes needless.

For the sake of comparison of methods let us (lastly) consider the funct.:

$$dy = \frac{x^n \, dx}{\sqrt{(2ax - x^2)}}$$

($n$ being any integer) which is commonly also treated by a formula of reduction, as is well known. We take in connection with this function one of our primitive forms, viz.:

$$dz = \frac{dx}{\sqrt{(2ax - x^2)}};\ \therefore\ z = \operatorname{arc\,sin}\frac{x - a}{a};\ x = a(1 + \sin z);$$

$$\therefore\ \frac{dy}{dz} = x^n = a^n(1 + \sin z)^n;$$

$$\therefore\ y = a^n\!\int(1 + \sin z)^n\, dz,$$

which evidently comes back to the forms just treated and is integrable without any difficulty.

We now proceed to the application of our elementary integrals to other forms not commonly found in the books.

Let it be required to integrate

$$dy = \frac{dx}{\sqrt{(1 - x^4)}\,[\sqrt{(1 + x^2)} + x\sqrt{2}]}.$$

This function suggests the elementary form

$$dz = \frac{\sqrt{2}.dx}{(1 + x^2)\sqrt{(1 - x^2)}} = \frac{\sqrt{2}.dx}{\sqrt{(1 - x^4)}\sqrt{(1 + x^2)}},$$

which gives

$$z = \operatorname{arc\,sin}\sqrt{\frac{1 - x^2}{1 + x^2}};\ \therefore\ \sin z = \sqrt{\frac{1 - x^2}{1 + x^2}};\ \cos z = x\sqrt{\frac{2}{1 + x^2}}.$$

Now $dy$ may be written thus

$$dy = \frac{dx\sqrt{(1 + x^2)}}{(1 + x^2)\sqrt{(1 - x^2)}[\sqrt{(1 + x^2)} + x\sqrt{2}]}$$

$$= \frac{dx}{(1 + x^2)\sqrt{(1 - x^2)}\left(1 + \frac{x\sqrt{2}}{\sqrt{(1 + x^2)}}\right)};$$

$$\therefore\ dy = \frac{1}{\sqrt{2}}.\frac{dz}{1 + \cos z};\ \therefore\ y = \tfrac{1}{2}\sqrt{2}.\operatorname{tg}\tfrac{1}{2}z;$$

$$\therefore\ \int\frac{dx}{\sqrt{(1 - x^4)}[\sqrt{(1 + x^2)} + x\sqrt{2}]} = \frac{1}{\sqrt{2}}\operatorname{tg}\left(\tfrac{1}{2}\operatorname{arc\,sin}\sqrt{\frac{1 - x^2}{1 + x^2}}\right).$$

Let it now be required to integrate the similar, but more general form:

$$dy = \frac{dx}{\sqrt{(1 - x^4)}[a\sqrt{(1 + x^2)} + c\,x]}.$$

Here, putting $c = b\sqrt{2}$, we can write

$$dy = \frac{dx}{\sqrt{(1 - x^4)}\sqrt{(1 + x^2)}\left(a + \frac{bx\sqrt{2}}{\sqrt{(1 + x^2)}}\right)}.$$

Again comparing with

$$dz = \frac{\sqrt{2}.dx}{\sqrt{(1-x^4)}\sqrt{(1+x^2)}};\ \therefore\ z = \text{arc}\sin\sqrt{\frac{1-x^2}{1+x^2}};\ \cos z = x\sqrt{\frac{2}{1+x^2}}$$

we have, when $a^2 > b^2$,

$$dy = \frac{1}{\sqrt{2}}\,\frac{dz}{a+b\cos z};\ \therefore\ y = \frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{(a^2-b^2)}}.\text{arc}\cos\left(\frac{b+a\cos z}{a+b\cos z}\right)$$

$$= \frac{1}{\sqrt{(2a^2-c^2)}}.\text{arc}\cos\left(\frac{c+a\sqrt{2}.\cos z}{a\sqrt{2}+c.\cos z}\right);$$

$$\therefore \int\frac{dx}{\sqrt{(1-x^4)}[a\sqrt{(1+x^2)}+cx]} = \frac{1}{\sqrt{(2a^2-c^2)}}$$

$$\times\ \text{arc}\cos\left(\frac{c+a\sqrt{2}.x\sqrt{[2\div(1+x^2)]}}{a\sqrt{2}+c.x\sqrt{[2\div(1+x^2)]}}\right)$$

$$= \frac{1}{\sqrt{(2a^2-c^2)}}.\text{arc}\cos\left(\frac{c\sqrt{(1+x^2)}+2ax}{a\sqrt{2}.\sqrt{(1+x^2)}+c\sqrt{2}.x}\right),$$

when $2a^2 > c^2$. In case $2a^2 < c^2$, we have:

$$\int\frac{dx}{\sqrt{(1-x^4)}[a\sqrt{(1+x^2)}+cx]} = \frac{1}{\sqrt{(c^2-2a^2)}}$$

$$\times\lg\left(\frac{2a+c\sqrt{(1+x^2)}+\sqrt{(c^2-2a^2)}\sqrt{(1-x^2)}}{a\sqrt{2}.\sqrt{(1+x^2)}+c.x}\right).$$

It is now readily seen that the method just employed amounts to the substitution of a new variable, for we have integrated in terms of $z$ what was given in terms of $x$. Hence all functions integrable in terms of $z$ become integrable also in terms of $x$. Thus, taking again

$$dz = \frac{\sqrt{2}.dx}{\sqrt{(1-x^2)(1+x^2)}};\ \therefore\ z = \text{arc}\sin\sqrt{\frac{1-x^2}{1+x^2}};\ \therefore\ \sin z = \sqrt{\frac{1-x^2}{1+x^2}};$$

$\therefore$ $\cos z = x\sqrt{2}\div\sqrt{(1+x^2)}$, let it be required to integrate in terms of $x$.

$$dy = \frac{dz}{(a+b\cos z)^2};\ \therefore\ y = \frac{b\sin z}{(b^2-a^2)(a+b\cos z)} - \frac{a}{b^2-a^2}\int\frac{dz}{a+b\cos z}.$$

This gives $dy = \dfrac{\sqrt{2}.dx}{\sqrt{(1-x^2)}\,(1+x^2)\,[a+bx\sqrt{2}\div\sqrt{(1+x^2)}]^2}$

$$= \frac{\sqrt{2}.dx}{\sqrt{(1-x^2)}\,[a^2+(2b^2+a^2)x^2+2\sqrt{2}.abx\sqrt{(1+x^2)}]};$$

$$\therefore \int\frac{dx}{\sqrt{(1-x^2)}[a^2+(2b^2+a^2)x^2+2\sqrt{2}.abx\sqrt{(1+x^2)}]} = \frac{1}{\sqrt{2}}\times$$

$$\frac{b\sqrt{(1-x^2)}}{(b^2-a^2)[a\sqrt{(1+x^2)}+b\sqrt{2}.x]} + \frac{1}{\sqrt{2}}\frac{a}{(a^2-b^2)^{\frac{3}{2}}}.\text{arc}\cos\left(\frac{b\sqrt{(1+x^2)}+a\sqrt{2}.x}{a\sqrt{(1+x^2)}+b\sqrt{2}.x}\right).$$

In this manner the function $z$ which we have here chosen will lead to a considerable number of new integrals, especially when the trigonometric functions in terms of $z$ are taken and expressed in terms of $x$, as above.

[To be continued.]